A CLASS OF EXACTLY SOLVABLE SCHRÖDINGER EQUATIONS

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Dedicated to Professor Josef Paldus on the occasion of his 70th birthday.

An algebraic approach to solving a class of one-particle Schrödinger equations is presented. As an example, quasi-exact solutions of the eigenvalue problem of a Hamiltonian describing two interacting particles confined in a parabolic well are obtained. This example constitutes a unification and a generalization of several models known in the literature, as the ones of Taut (*Phys. Rev. A* **1993**, *48*, 3561) and of Samanta and Ghosh (*Phys. Rev. A* **1990**, *42*, 1178). Two confined particles interacting by Coulomb forces and the nuclear motion of a diatomic molecule are discussed as practical implementations.

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Analytically solvable models supply most valuable opportunities for checking the correctness and range of applicability of many approximate methods. In particular, many quantum-chemical methods have been derived, in a more or less straightforward way, from analytically solvable models such as the hydrogen-like atom or the harmonic oscillator. During the last two decades several new quasi-analytically solvable models have been introduced to quantum chemistry and to atomic and molecular physics[1–14]. Probably the best known example is a system composed of two particles confined in a harmonic oscillator potential and interacting by the Coulomb force[1–5]. The corresponding Hamiltonian eigenvalue problem is analytically solvable if the harmonic oscillator and the Coulomb potential coupling constants are related in a specific way. This system, referred to as harmonium, discovered in 1968 by Santos[1] and then rediscovered by Taut[2] and by other authors[3], has been used in density functional theory[6,7], electron correlation[8], and quantum dot[3] studies.

By introducing an additional linear term to the interaction potential and, consequently, an additional coupling constant to the Hamiltonian[9,10], one

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obtains a larger set of analytical solutions of the corresponding eigenvalue problem. Most interesting are successful extensions of this approach to two-center problems[11,12]. Finally, generalizations for more-than-two-particle systems do not seem to be hopeless[13,14].

The inspiration for this work has been taken from a paper by Bose and Gupta[15] where exact solutions of the one-dimensional Schrödinger equation for a large number of potentials were obtained. Here we concentrate on the method of solving the Hamiltonian eigenvalue equation rather than on extending the set of quasi-exactly solvable problems. In particular, by formulation of a simple inverse problem we can generate a large number of potentials for which the corresponding Schrödinger equation is quasi-exactly solvable. Also the method of solving the equation has been simplified. The conditions for the exact solubility have been transformed to an eigenvalue problem of a symmetric tridiagonal matrix. In effect several new features of the analytical solutions have been observed. The method presented may be used, in a rather obvious way, to construct new analytically solvable models.

The exact solutions appear to possess several remarkable properties. In particular they are associated with some additional degeneracies in the eigenvalue spectrum. Therefore, one should expect that there exists a hidden symmetry responsible for the existence of analytical solutions. Work in this direction is in progress. In this context one should mention a brilliant review due to Adams, Čížek and Paldus[16] in which applications of dynamic groups to solving Hamiltonian eigenvalue problems are presented.

FORMULATION OF THE PROBLEM

Let us consider two particles with masses m_1 and m_2 interacting by a potential

$$\mathrm{V}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \mathrm{V}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) = \mathrm{V}(r) \tag{1}$$

which depends only on the distance

$$r = |\boldsymbol{r}| = |\boldsymbol{r}_1 - \boldsymbol{r}_2| \tag{2}$$

between the particles. In order to allow for a discrete energy spectrum, either the potential has to be attractive in a certain range of r or the motion of these particles has to be confined. Let $\mathrm{U}(\boldsymbol{r}_1,\boldsymbol{r}_2)$ be the confining potential. The most interesting cases with U composed of Coulomb potentials and $\mathrm{V} = 1/r$ correspond to helium-like atoms or to two-electron molecules. Unfortunately, the resulting Schrödinger equation is non-separable in these cases. Probably, the only confinement which allows for the separability is the harmonic one:

$$\mathrm{U} = \frac{\omega^2}{2}(m_1\boldsymbol{r}_1^{\,2} + m_2\boldsymbol{r}_2^{\,2}) = \frac{\omega^2}{2}(M\boldsymbol{R}^2 + \mu\boldsymbol{r}^2) \tag{3}$$

where

$$M = m_1 + m_2\ , \quad \mu = \frac{m_1 m_2}{M} \tag{4}$$

and

$$\boldsymbol{R} = \frac{m_1}{M}\boldsymbol{r}_1 + \frac{m_2}{M}\boldsymbol{r}_2\ . \tag{5}$$

In such case the Hamiltonian for two confined particles

$$\mathrm{H}(\boldsymbol{r}_1,\boldsymbol{r}_2) = \frac{\mathrm{p}_1^{\,2}}{2m_1} + \frac{\mathrm{p}_2^{\,2}}{2m_2} + \mathrm{U}(\boldsymbol{r}_1,\boldsymbol{r}_2) + \mathrm{V}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) \tag{6}$$

can be expressed as sum of two Hamiltonians, $\mathrm{H}_\mathrm{R}(\boldsymbol{R})$ describing the center of mass and $\mathrm{H}_\mathrm{r}(\boldsymbol{r})$ describing the relative motion of the particles. The center of mass Hamiltonian

$$\mathrm{H}_\mathrm{R}(\boldsymbol{R}) = -\frac{1}{2M}\Delta_\mathrm{R} + \frac{M\omega^2}{2}R^2 \tag{7}$$

describes free motion of a particle of mass M if $\omega = 0$, and corresponds to a three-dimensional isotropic harmonic oscillator if $\omega > 0$. The Hamiltonian that describes the relative motion reads

$$\mathrm{H}_\mathrm{r}(\boldsymbol{r}) = -\frac{1}{2\mu}\Delta_\mathrm{r} + \frac{\mu\omega^2}{2}r^2 + \mathrm{V}(r)\ . \tag{8}$$

Then, the solutions of the two-particle eigenvalue problem

$$\mathrm{H}(\boldsymbol{r}_1,\boldsymbol{r}_2)\,\Psi(1,2) = E_{nl;\,\nu\lambda}\,\Psi(1,2) \tag{9}$$

can be expressed as

$$\Psi(1,2) = \Xi(\boldsymbol{R})_{\nu\lambda\mu}\,\Phi(\boldsymbol{r})_{nlm}\,\Theta_{sm_s} \tag{10}$$

where Θ_{sm_s} is a two-particle spin function,

$$\mathrm{H}_{\mathrm{R}}\,\Xi(\boldsymbol{R})_{\nu\lambda\mu} = E^{R}_{\nu\lambda}\,\Xi(\boldsymbol{R})_{\nu\lambda\mu} \tag{11}$$

is the center of the mass eigenvalue problem and

$$\mathrm{H}_{\mathrm{r}}\Phi(\boldsymbol{r})_{nlm} = E_{nl}\,\Phi(\boldsymbol{r})_{nlm} \tag{12}$$

describes the relative motion. Equations (*11*) and (*12*) are spherically symmetric. Therefore

$$\Xi(\boldsymbol{R})_{\nu\lambda\mu} = \frac{1}{R}\,\xi(R)_{\nu\lambda}\,Y_{\lambda\mu}(\hat{\boldsymbol{R}}) \tag{13}$$

and

$$\Phi(\boldsymbol{r})_{nlm} = \frac{1}{r}\,\phi(r)_{nl}\,Y_{lm}(\hat{\boldsymbol{r}}) \tag{14}$$

where $\hat{\boldsymbol{R}} = \boldsymbol{R}/R$ and $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. The relative motion equation

$$\left[-\frac{1}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{2\mu r^2} + \mathrm{W}(r)\right]\phi(r)_{nl} = E_{nl}\phi(r)_{nl} \tag{15}$$

where

$$\mathrm{W}(r) = \frac{\mu\omega^2}{2}r^2 + \mathrm{V}(r)$$

may describe a large number of systems such as:

– Hook atom (harmonium), i.e. two electrons in a parabolic confinement.

In this case $\mu = m/2$ and

$$\mathsf{V} = \frac{\zeta}{r} \tag{16}$$

with $\zeta = 1$.

– Confined positronium – the same as before, but $\zeta = -1$.

– Two atoms (neutral or ionized) or a charged particle and an atom confined in a parabolic well. The interaction is then described by an effective (not necessarily Coulomb) potential.

– Nuclear motion of a diatomic molecule. In this case

$$\mathsf{W} = \frac{\mu\omega^2}{2}(r - r_\mathrm{e})^2 \tag{17}$$

where r_e is the nuclear equilibrium distance, corresponds to the harmonic approximation.

– Two particles (e.g. quarks) interacting by non-electromagnetic forces.

The relative motion can also be interpreted as described by an effective Hamiltonian for a quasi-particle of mass μ confined in an external potential. In particular, by properly selecting V we can get models of centrally confined atoms.

THE EXPONENTIAL ANSATZ

Recently Bose and Gupta[15] formulated the so called *exponential ansatz* method to find exact solution of a large class of Schrödinger equations. In their approach the exponential part of the wave function is selected for each kind of the interaction potential on an ad hoc basis. Here we present an alternative approach in which the potential for which an analytical solution of the Schrödinger equation exists is determined by a simple inverse-problem analysis.

Equation (*15*) can be rewritten as

$$\mathsf{W} - E = \frac{1}{\phi}\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2} \tag{18}$$

where, for simplicity, we set $\mu = 1/2$. We are looking for potentials V for which Eq. (*18*) is analytically solvable. Let us assume that the ground-state wave function is nodeless and can be expressed in the form

$$\phi(r) \sim r^{l+1}(1+r/b)\mathrm{e}^{Q(r)} \tag{19}$$

where $b > 0$ and Q has been chosen in such a way that $\phi(r)$ is normalizable. The substitution of the wave function (*19*) into Eq. (*18*) yields

$$\mathsf{W} - E = Q'' + Q'\left[Q' + \frac{2(l+1)}{r} + \frac{2}{b+r}\right] + \frac{2(l+1)}{r(b+r)} \,. \tag{20}$$

If the behavior of the wave function for $r \to \infty$ has to be determined by the quadratic term in Eq. (*15*), then the simplest form of the exponent in Eq. (*19*) is $Q = -\omega r^2/4$. Under this assumption Eq. (*20*) reads

$$\mathsf{W} - E = \frac{\omega^2 r^2}{4} + \frac{\zeta}{r} - \frac{\zeta - b\omega}{b+r} - \omega\left(l + \frac{5}{2}\right) \tag{21}$$

where

$$b = \frac{2(l+1)}{\zeta} \,. \tag{22}$$

As we see, the potential contains a Coulomb term and a "screening" correction. The parameters ζ and ω are arbitrary but their choice defines the "screening". An important special case of a pure Coulomb potential corresponds to $\omega = \zeta/b$. In this case

$$\mathsf{V} = \frac{\zeta}{r}, \quad \omega = \frac{\zeta^2}{2(l+1)}, \quad E = \zeta^2 \frac{2l+5}{4(l+1)} \,. \tag{23}$$

If $\zeta = 1$ then we have the well known Hook atom or harmonium[1–5].

A set of interesting special cases is obtained by putting $a_1 = 1/b = 0$. In particular, if $Q = -\omega(r - r_\mathrm{e})^2/4$, then

$$\mathsf{W} = \frac{\omega^2}{4}(r - r_\mathrm{e})^2 + \frac{\zeta}{r} \tag{24}$$

where $r_\mathrm{e} = \zeta/\omega(l+1)$. This case corresponds to the model introduced by Ghosh and Samantha[9,10]. It can also be used to describe rovibrational spectra of diatomic molecules. Taking $Q = -Ar^{q+1}/(q+1)$, where A is a positive constant, we get a family of anharmonic oscillators with

$$\mathsf{W} = A^2 r^{2q} - A(q+2l+2) r^{q-1} \ .$$

Many other examples can be constructed in a similar way.

So far we have discussed nodeless solutions of exactly solvable equations. In order to solve the eigenvalue problem for excited states let us replace the nodeless wave function (*19*) by

$$\phi(r) \sim r^{l+1} P(r) \, \mathrm{e}^{Q(r)} \tag{25}$$

where

$$P(r) = \sum_{i=0}^{p} a_i r^i \tag{26}$$

is a polynomial. Then Eq. (*15*) becomes

$$P'' + 2P'\left(\frac{l+1}{r} + Q'\right) + P(R - \mathsf{W} + E) = 0 \tag{27}$$

where

$$R = Q'' + Q'\left(Q' + \frac{2(l+1)}{r}\right).$$

Though, obviously, one can generate potentials corresponding to an arbitrarily chosen wave function, there is very limited, if any, interest in this sort of analysis. Instead, one can select potentials of interest, for which analytical ground-state solutions exist, and then find the polynomials corresponding to the excited states. As an example let us consider the potential given by Eq. (*24*).

By substitution of Eqs (*24*) and (*26*) into Eq. (*27*), we get

$$\begin{aligned} B_0 a_0 + C_1 a_1 &= 0 \\ A_i a_i + B_{i+1} a_{i+1} + C_{i+2} a_{i+2} &= 0 \, , \quad i = 0, 1, 2, \ldots, p-1 \end{aligned} \tag{28}$$

where

$$\begin{aligned} A_i &= E - \omega(i + l + 3/2) \\ B_i &= \omega r_{\mathrm{e}}(i + l + 1) - \zeta \\ C_i &= i(i + 2l + 1) \ . \end{aligned} \tag{29}$$

The recurrence terminates at a_p if $A_p = 0$, i.e. if

$$E = \omega(p + l + 3/2) . \qquad (30)$$

Thus, we have $A_i = \omega(p - i)$. Consequently, the recurrence relations (*29*) form a set of $p + 1$ linear equations

$$\mathbf{W}_{p+1}\boldsymbol{a} = 0 \qquad (31)$$

where

$$\mathbf{W}_{p+1} = \begin{pmatrix} B_0 & C_1 & 0 & \cdots & 0 & 0 & 0 \\ A_0 & B_1 & C_2 & \cdots & 0 & 0 & 0 \\ 0 & A_1 & B_2 & \cdots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & A_{p-2} & B_{p-1} & C_p \\ 0 & 0 & 0 & \cdots & 0 & A_{p-1} & B_p \end{pmatrix} \qquad (32)$$

and $\boldsymbol{a} = \{a_0, a_1, \dots, a_p\}$. The solutions exist if

$$|\mathbf{W}_{p+1}| = 0 . \qquad (33)$$

The determinant may be easily evaluated using simple recurrent formulas:

$$\begin{aligned} |\mathbf{W}_1| &= B_0 \\ |\mathbf{W}_2| &= B_1|\mathbf{W}_1| - A_0C_1 \\ |\mathbf{W}_i| &= B_{i-1}|\mathbf{W}_{i-1}| - A_{i-2}C_{i-1}|\mathbf{W}_{i-2}| , \quad i = 2, 3, \dots, p . \end{aligned} \qquad (34)$$

Equation (*30*) which defines the energy of the relative motion of two particles corresponding to an exact solution of the eigenvalue equation, results from the condition of the termination of the recurrence relation (*29*). Another condition, necessary for the existence of the polynomial P, is given by Eq. (*33*). The determinant $|\mathbf{W}|$ depends on three parameters: ω, ζ, and r_e (the fourth parameter μ has been, arbitrarily, set equal to 1/2). Due to the condition $|\mathbf{W}| = 0$ only two of these parameters are independent. In particular, in the case of harmonium $r_e = 0$ and $\zeta = 1$. Therefore for each pair (l,p)

we have a discrete set (composed of at most $p + 1$ elements) of ω values for which Eq. (*33*) is fulfilled and, consequently, exists an analytical solution of Eq. (*15*). In the model of Ghosh and Samantha[9,10], $\zeta = 1$ but for each value of ω there exists a specific r_e for which $|W| = 0$ and, thus, the Hamiltonian eigenvalue problem is analytically solvable. Finally, setting $\zeta = 0$ we get an equation modeling the nuclear motion in a diatomic molecule. Also here for each ω one can find r_e for which an analytical solution exists.

SYMMETRIZATION OF THE RECURRENT RELATIONS

The three-step recurrence relation (*28*) described by a tridiagonal system of linear equations (*31*) can be transformed to an eigenvalue problem of a symmetric tridiagonal matrix[5]. To this end let us rewrite Eqs (*31*) in the form

$$\tilde{W}_{p+1}\tilde{\boldsymbol{a}} = 0 \tag{35}$$

where

$$\tilde{W}_{p+1} = \boldsymbol{p} W_{p+1} \boldsymbol{q} \ , \tag{36}$$

$\tilde{\boldsymbol{a}} = \boldsymbol{q}^{-1}\boldsymbol{a}$, and $\boldsymbol{p}$, $\boldsymbol{q}$ are diagonal matrices with elements $p_{ik} = p_i\delta_{ik}$, $q_{ik} = q_i\delta_{ik}$, i, $k = 0, 1, 2, \ldots, p$. If we set

$$p_0 = 1 , \quad p_i = \sqrt{\frac{C_i}{A_{i-1}}\frac{B_{i-1}}{B_i}}\, p_{i-1} \tag{37}$$

and

$$q_0 = 1 , \quad q_i = \sqrt{\frac{A_{i-1}}{C_i}\frac{B_{i-1}}{B_i}}\, q_{i-1} \tag{38}$$

then

$$\tilde{W}_{p+1} = \begin{pmatrix} -s & D_1 & 0 & \cdots & 0 & 0 \\ D_1 & -s & D_2 & \cdots & 0 & 0 \\ 0 & D_2 & -s & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \cdots & -s & D_p \\ 0 & 0 & 0 & \cdots & D_p & -s \end{pmatrix} \tag{39}$$

and Eq. (*35*) may be written as an eigenvalue problem of a symmetric tridiagonal matrix:

$$(\boldsymbol{T} - s\boldsymbol{I})\,\tilde{\boldsymbol{a}} = 0 \qquad (40)$$

where $\boldsymbol{I}$ is a unit matrix,

$$s = -\frac{B_0}{c}$$

and

$$T_{ik} = D_k \delta_{i+1,k} + D_i \delta_{i,k+1}\,, \quad i, k = 0, 1, 2, \ldots, p$$

with

$$D_i = \frac{1}{c}\sqrt{\frac{C_i A_{i-1}}{\mathcal{B}_i \mathcal{B}_{i-1}}}\,, \quad i = 1, 2, \ldots, p \qquad (41)$$

and $\mathcal{B}_i = B_i / B_0$. The constant c is, in principle, arbitrary. It has been introduced for convenience since its proper choice may simplify the formulas. The secular equation (*33*) is then replaced by

$$|\tilde{W}_{p+1}| = 0 \qquad (42)$$

where, according to Eqs (*36*)–(*38*),

$$|\tilde{W}_{p+1}| = C\,|W_{p+1}| \qquad (43)$$

with

$$C = \prod_{i=0}^{p} p_i q_i = \prod_{i=0}^{p} \frac{1}{\mathcal{B}_i}\,.$$

The recurrent relations (*35*) may be rewritten as

$$\begin{aligned} \tilde{W}_1 &= -s \\ \tilde{W}_2 &= -s\tilde{W}_1 - D_1^2 \\ \tilde{W}_i &= -s\tilde{W}_{i-1} - D_{i-1}^2 \tilde{W}_{i-2}\,, \quad i = 2, 3, \ldots, p+1\,. \end{aligned} \qquad (44)$$

The first five determinants are given by:

$$\begin{aligned}
\tilde{W}_1 &= -s \\
\tilde{W}_2 &= s^2 - D_1^2 \\
\tilde{W}_3 &= -s[s^2 - (D_1^2 + D_2^2)] \\
\tilde{W}_4 &= s^4 - s^2(D_1^2 + D_2^2 + D_3^2) + D_1^2 D_3^2 \\
\tilde{W}_5 &= -s[s^4 - s^2(D_1^2 + D_2^2 + D_3^2 + D_4^2) + (D_1^2 D_3^2 + D_1^2 D_4^2 + D_2^2 D_4^2)] .
\end{aligned} \tag{45}$$

The matrix $\boldsymbol{T}$ is traceless and, as one can easily show, its eigenvalue spectrum is symmetric, i.e. if $s_0 \leq s_1 \leq s_2 \leq \ldots \leq s_p$ then $s_j = -s_{p-j}$, $j = 1, 2, \ldots, (p+1)/2$ and, if p is even (i.e. the dimension of $\boldsymbol{T}$ is odd), then $s_{p/2} = 0$. The case of $s = 0$ has to be treated with a special care since it implies $B_0 = 0$ and, thus, the values of D_i may be ill-defined. This point is discussed in more detail in the next section.

EXAMPLES

In this section we consider several exactly solvable models derived from the potential composed of a non-central spherical harmonic oscillator term and a Coulomb term (*24*).

Harmonium and Confined Positronium

If $r_e = 0$, then potential (*24*) describes two confined particles with either repulsive (if $\zeta > 0$) or attractive (if $\zeta < 0$) Coulomb interaction. In particular, $\zeta = 1$ corresponds to harmonium and $\zeta = -1$ to a confined positronium. Taking $c = \sqrt{\omega}$ we have

$$D_i^2 = i(i + 2l + 1)(p - i + 1) \tag{46}$$

and $s = \zeta / \sqrt{\omega}$. Then, the positive eigenvalues correspond to a repulsive potential V (to harmonium if $\zeta = 1$) and the negative ones – to an attractive one (to confined positronium if $\zeta = -1$). The case of $s = 0$ corresponds to $\zeta = 0$, i.e. to two confined non-interacting particles. Since the spectrum $\{s_0, s_1, \ldots, s_p\}$ of $\boldsymbol{T}$ is symmetric, the energies which correspond to the analytically solvable cases, for given $|\zeta|$, l and p, are the same for the attractive and for the repulsive potential. Upon the substitution $\rho = \sqrt{\omega} r$ and $\mu = 1/2$ Eq. (*15*) for the interaction potential (*16*) reads

$$\left[-\frac{d^2}{d\rho^2} + \frac{l(l+1)}{\rho^2} + \frac{1}{4}\rho^2 \pm \frac{|s|}{\rho}\right]\phi^{\pm}(\rho) = \varepsilon^{\pm}\phi^{\pm}(\rho) \tag{47}$$

where $\varepsilon = E/\omega$. In analytically solvable cases $\varepsilon^+ = \varepsilon^-$ and, consequently, the corresponding eigenfunctions are related as $\phi^+(\rho) = \phi^-(-\rho)$. Then, the analytical eigenfunctions (*25*) for the repulsive potential differ from their counterparts for the attractive one by the sign of the odd powers of r in the polynomial P, i.e.

$$\phi(\rho) = \rho^{l+1} P^{\pm}(\rho)\, e^{-\rho^2/4} \tag{48}$$

where

$$P^{+}(\rho) = \sum_{i=0}^{p} \tilde{a}_i \rho^i$$

$$P^{-}(\rho) = \sum_{i=0}^{p} (-1)^i \tilde{a}_i \rho^i \tag{49}$$

and $\tilde{a}_i = a_i/\omega^{i/2}$. Of course, for a solution, which may be obtained by numerical integration of Eq. (*47*) for arbitrary values of ω and ζ, these conditions are, in general, not fulfilled. As an example plot of spectra of harmonium ($\zeta = 1$) and of the confined positronium ($\zeta = -1$), versus log ω, is presented in Fig. 1. The spectra are scaled by the excitation energy to the 10th excited state of the confined positronium, i.e. the quantities plotted are equal to

$$\Delta_N = \frac{E_N - E_1^{\mathrm{ps}}}{E_{10}^{\mathrm{ps}} - E_1^{\mathrm{ps}}} \tag{50}$$

where the superscript ps refers to the confined positronium and E_N is the energy of the N-th energy level of either harmonium or confined positronium. The energies have been obtained by numerical integration of Eq. (*47*)[5].

If $p = 1$ then $D_1^2 = 2(l + 1)$. Hence,

$$s = \frac{\zeta}{\sqrt{\omega}} = \pm\sqrt{2(l+1)} \tag{51}$$

and, according to Eq. (*30*),

$$E = \omega\left(l + \frac{5}{2}\right) = \zeta^2 \, \frac{2l+5}{4(l+1)} \, .$$

Assuming normalization $a_0 = 1$, the polynomials in the corresponding wave functions (*50*) are $P^{\pm} = 1 + a_1 r$, where

$$a_1 = \frac{\zeta}{2(l+1)} = \pm\sqrt{\frac{\omega}{2(l+1)}} \, .$$

If $a_1 > 0$, i.e. if V is repulsive, as it is in the case of harmonium, the wave function is nodeless and corresponds to the ground state. In the opposite case (e.g. in the case of confined positronium), the wave function has

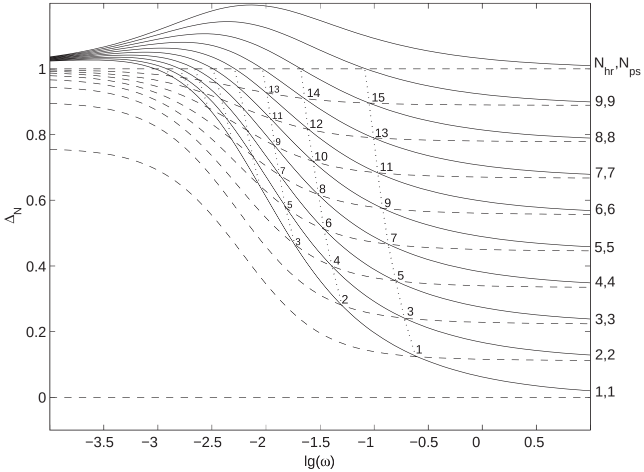

FIG. 1

Energy levels of harmonium (solid lines) and of the confined positronium (broken lines) for l = 0 scaled by the excitation energy to the 10th excited state of the confined positronium, versus lg ω. The points where the energy levels cross correspond to the analytical solutions with $\zeta = \pm 1$ and are labeled by the corresponding values of p. At the right-hand side the consecutive numbers of the energy levels in the spectra of harmonium (N_{hr}) and of the confined positronium (N_{ps}) are given. The points corresponding to $N_{ps} - N_{hr} = n_0$ are connected by dotted lines (the rightmost line corresponds to $n_0 = 1$, and the consecutive ones to $n_0 = 2, 3, \ldots$).

one node (at $r_0 = \sqrt{2(l+1)/\omega}$) and corresponds to the first excited state (cf. Fig. 1).

A similar analysis for $p = 2$ gives

$$s = \frac{\zeta}{\sqrt{\omega}} = \pm\sqrt{2(4l+5)}$$

$$E = \omega\left(l + \frac{7}{2}\right) = \zeta^2 \; \frac{2l+7}{4(4l+5)}$$

$$a_1 = \frac{\zeta}{2(l+1)} = \pm\frac{\sqrt{2\omega(4l+5)}}{2(l+1)}$$

$$a_2 = \frac{\zeta^2}{4(l+1)(4l+5)} = \frac{\omega}{2(l+1)}$$

where $P^{\pm} = 1 + a_1 r + a_2 r^2$. Here again, for a repulsive V, the wave function is nodeless and corresponds to the ground state while in the opposite case it has two nodes (at $r_0^{\pm} = (\sqrt{4l+5} \pm 1)/\sqrt{2\omega}$) and describes the second excited state.

Model of Ghosh and Samantha

In this model we consider two particles confined by a harmonic oscillator potential and interacting by a Coulomb potential with an additional linear term[9,10], i.e.

$$\mathrm{V} = \frac{\zeta}{r} - ar \; . \tag{52}$$

This model is described by potential (*24*) if we set $2a = r_e\,\omega^2$ and absorb the constant term $r_e^2\omega^2/4$ in the energy. In this case

$$s = -\frac{B_0}{c} = \frac{2a}{\omega c}(\tau - l - 1) \tag{53}$$

where $\tau = \zeta\omega/2a$. Setting $s = 0$ implies $\tau = l + 1$, i.e.

$$a = \frac{\zeta\omega}{2(l+1)} .$$

In particular, for $p = 0$, we get

$$E = \omega\left(l + \frac{3}{2}\right) - \frac{a^2}{\omega^2} = \frac{a}{\zeta}(2l+3)(l+1) - \frac{\zeta^2}{4(l+1)^2} \tag{54}$$

and

$$\phi = r^{l+1} e^{-\omega r^2/4 + ar/\omega} .$$

If $B_0 \neq 0$, i.e. $\tau \neq l + 1$, we set $c = \sqrt{\omega}(\tau - l - 1)$. Then

$$s^2 = \frac{4a^2}{\omega^3} = \frac{\zeta^2}{\tau^2\omega} \tag{55}$$

and

$$D_i^2 = \frac{i(i + 2l + 1)(p - i + 1)}{(\tau - i - l - 1)(\tau - i - l)} . \tag{56}$$

The right-hand side of Eq. (*56*) has to be positive for $i = 1, 2, ..., p$. Therefore, the values of τ are restricted to either $\tau < l + 1$ or $\tau > p + l + 1$. The cases of $\tau = l + 1$ and $\tau = p + l + 1$ correspond, respectively, to $B_0 = 0$ and to $B_p = 0$. As a consequence, in both cases, the degree of the secular equation (*33*) is reduced from $p + 1$ to $p - 1$.

In the case of $p = 1$, according to Eq. (*45*), analytical solutions exist if

$$s^2 = D_1^2 \tag{57}$$

i.e. if

$$\frac{\zeta^2}{\omega} = \frac{2(l+1)}{[1-(l+2)/\tau][1-(l+1)/\tau]} . \tag{58}$$

At the limit of $a = 0$, i.e. $\tau \to \infty$, the model is reduced to the case discussed in the preceding subsection and Eq. (*58*) is reduced to Eq. (*51*). Alternatively, Eq. (*57*) can be written as

$$\frac{4a^2}{\omega^3} = \frac{2(l+1)}{(\tau - l - 2)(\tau - l - 1)} . \tag{59}$$

As one can see, for each value of τ, except for $l + 1 < \tau < l + 2$, one can find a value of ζ^2/ω (according to Eq. (*58*)) or a^2/ω^3 (according to Eq. (*59*)) such that Eq. (*33*) (or equivalently (*42*)) is fulfilled. This statement may be generalized for the case of an arbitrary p with values of $\tau \notin \langle l + 1, p + l + 1\rangle$.

Diatomic Molecule

The potential (*24*) may describe the relative motion of the nuclei of a diatomic molecule in the harmonic approximation (if $\zeta = 0$) or corrected for the nuclear repulsion (if $\zeta > 0$). Except for the physical interpretation, this case is very similar to the previous one. Now

$$s = -\frac{B_0}{c} = \frac{r_\mathrm{e}\omega}{c}(l + 1 - t) \qquad (60)$$

where $t = \zeta/r_\mathrm{e}\,\omega$. If $s = 0$ then $t = l + 1$, i.e. $\zeta = r_\mathrm{e}\,\omega(l + 1)$. In particular, if $p = 0$, then

$$E = \omega\left(l + \frac{3}{2}\right) = -\frac{\zeta(2l + 3)}{2r_\mathrm{e}(l + 1)} \qquad (61)$$

and $\phi \sim r^{l+1}\mathrm{e}^{-\omega(r-r_\mathrm{e})^2/4}$. If $B_0 \neq 0$, we set $c = \sqrt{\omega}(l + 1 - t)$. Then

$$s = r_\mathrm{e}\sqrt{\omega} \qquad (62)$$

and

$$D_i^2 = \frac{i(i + 2l + 1)(p - i + 1)}{(i + l + 1 - t)(i + l - t)}\,. \qquad (63)$$

Similarly as in the case of Ghosh and Samantha model, for each value of $t \notin \langle l + 1, p + l + 1\rangle$ and ω, one can find such r_e that the secular equation (*42*) is fulfilled. Therefore for a given pair $\{\zeta,\omega\}$ there exists r_e for which the corresponding Schrödinger equation is analytically solvable. In particular, the plain harmonic approximation corresponds to $t = 0$. As it was mentioned before, for a given t spectrum $\{s_0, s_1, \ldots, s_p\}$ of $\boldsymbol{T}$ (Eq. (*40*)) is symmetric. Then, according to Eq. (*62*), analytical solutions exist for pairs $\pm|r_\mathrm{e}|$. Besides, since t has to be fixed, if the analytical solution corresponding to $|r_\mathrm{e}|$ is associated with ζ, then the one corresponding to $-|r_\mathrm{e}|$ has to be associated with

–ζ. However, in the case of a diatomic molecule, ζ ≥ 0 and therefore only solutions associated with a given sign of r_e are physically acceptable (except for the case of ζ = 0).

FINAL REMARKS

Analytical solutions of the eigenvalue problem of the Schrödinger Hamiltonian describing several systems of two interacting particles have been obtained using a simple algebraic method. The analytical solutions exist only for a specific set of the coupling constants defining the Hamiltonian. Besides, only a subset of all solutions corresponding to these coupling constants is expressible in the analytical form. The analytical solutions exhibit remarkable properties by which they differ in an essential way from the other solutions. In particular, in the case of the analytical solutions, some specific degeneracies appear in the eigenvalue spectrum. These properties strongly suggest that a hidden dynamic symmetry of these systems is responsible for the existence of the analytical solutions.

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